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LETTER TO THE EDITOR

Conformally covariant energy–momentum tensor for spin 2

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Abstract. We derive the expression of the conformally covariant energy–momentum tensor for spin 2.

By starting from the conformally covariant Lagrangian, we derived a general expression of the conformally covariant energy–momentum tensor for spin 0, 1 and $\frac{1}{2}$ (Xu 1981a,b). In this letter we shall discuss the case for the spin-2 tensor field.

The massless spin-2 tensor field denoted by symmetry tensor $h_{\mu\nu}$ transforms under the special conformal transformations as (Isham *et al* 1970, Barut and Xu 1982)

$$h'_{\mu\nu} = \Omega[g^{\alpha}_{\mu}g^{\beta}_{\nu} + (c^{\lambda}x^{\sigma} - x^{\lambda}c^{\sigma})(I_{\lambda\sigma})_{\mu\nu}{}^{\alpha\beta}]h_{\alpha\beta} \tag{1}$$

where

$$(I_{\lambda\sigma})_{\mu\nu}{}^{\alpha\beta} = (g_{\lambda\mu}g_{\sigma}{}^{\alpha} - g_{\lambda}{}^{\alpha}g_{\sigma\mu})g_{\nu}{}^{\beta} + (g_{\lambda\nu}g_{\sigma}{}^{\beta} - g_{\lambda}{}^{\beta}g_{\sigma\nu})g_{\mu}{}^{\alpha} \tag{2}$$

and

$$\Omega = 1 + 2c^{\mu}x_{\mu} + c^2x^2. \tag{3}$$

It is known that the conformally covariant Lagrangian of $h_{\mu\nu}$ has the following form (Barut and Xu 1982)

$$\mathscr{L} = -\tfrac{1}{2}(\partial_{\sigma}h_{\mu\nu})^2 + \tfrac{2}{3}\partial_{\sigma}h_{\mu\nu}\partial^{\nu}h^{\mu\sigma} + \tfrac{1}{6}(\partial_{\sigma}h)^2 - \tfrac{1}{3}\partial_{\sigma}h\,\partial_{\mu}h^{\mu\sigma} \tag{4}$$

where $h \equiv h_{\mu}{}^{\mu}$. Under special conformal transformations, equation (3) becomes

$$\mathscr{L}' = \Omega^4\mathscr{L} + 2c^{\lambda}R_{\lambda} \tag{5}$$

$$R_{\lambda} = \partial^{\sigma}R_{\sigma\lambda}$$

$$R_{\sigma\lambda} = \tfrac{1}{3}[\tfrac{1}{2}g_{\sigma\lambda}(h_{\mu\nu})^2 + 2h_{\sigma\nu}h_{\lambda}{}^{\nu} - hh_{\sigma\lambda}] \tag{6}$$

so both $\mathscr{L}$ and $\mathscr{L}'$ lead to the same conformally convariant equation of motion

$$\partial^2h_{\mu\nu} - \tfrac{2}{3}(\partial_{\mu}\partial^{\sigma}h_{\nu\sigma} + \partial_{\nu}\partial^{\sigma}h_{\mu\sigma}) + \tfrac{1}{3}\partial_{\mu}\partial_{\nu}h + \tfrac{1}{3}g_{\mu\nu}(\partial^{\lambda}\partial^{\sigma}h_{\lambda\sigma} - \partial^2h) = 0. \tag{7}$$

We now derive the conformally covariant energy–momentum tensor $\theta_{\mu\nu}$ of $h_{\mu\nu}$ using the same approach as previously. The resultant expression of $\theta_{\mu\nu}$ is

$$\begin{aligned}\theta_{\mu\nu} = T_{\mu\nu} &- \tfrac{1}{2}\partial^{\lambda}\{[\pi_{\lambda}{}^{\alpha\beta}(I_{\mu\nu})_{\alpha\beta}{}^{\sigma\rho} + \pi_{\mu}{}^{\alpha\beta}(I_{\nu\lambda})_{\alpha\beta}{}^{\sigma\rho} + \pi_{\nu}{}^{\alpha\beta}(I_{\mu\lambda})_{\alpha\beta}{}^{\sigma\rho}]h_{\sigma\rho}\}\\ &- \tfrac{1}{2}\partial^2R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\partial^{\lambda}\partial^{\rho}R_{\lambda\rho} + \tfrac{1}{2}\partial_{\mu}\partial^{\rho}R_{\rho\nu}\\ &+ \tfrac{1}{2}\partial_{\nu}\partial^{\rho}R_{\rho\mu} - \tfrac{1}{6}(\partial_{\mu}\partial_{\nu} - g_{\mu\nu}\partial^2)R_{\lambda}{}^{\lambda}\end{aligned} \tag{8}$$

where

$$T_{\mu\nu} = g_{\mu\nu}\mathscr{L} - \pi_\mu{}^{\alpha\beta}\partial_\nu h_{\alpha\beta} \tag{9}$$

and

$$\pi_\mu{}^{\alpha\beta} = \partial\mathscr{L}/\partial\partial^\mu h_{\alpha\beta}. \tag{10}$$

From (4), (6), (9) and (10), equation (8) can be rewritten in an explicit form

$$\begin{aligned}\theta_{\mu\nu} = g_{\mu\nu}[&-\tfrac{7}{18}(\partial_\lambda h_{\alpha\beta})^2 - \tfrac{1}{3}(\partial^\alpha h_{\alpha\beta})^2 + \tfrac{1}{3}\partial^\lambda h^{\alpha\beta}\partial_\alpha h_{\beta\lambda}\\
&+\tfrac{1}{18}(\partial_\lambda h)^2 + \tfrac{1}{3}\partial^\alpha h\partial^\beta h_{\beta\alpha}]\\
&+\tfrac{2}{3}\partial^\alpha h_\mu{}^\beta\partial_\alpha h_{\nu\beta} + \tfrac{4}{3}\partial^\alpha h_\mu{}^\beta\partial_\beta h_{\nu\alpha} - \tfrac{4}{3}\partial^\alpha h_{\mu\nu}\partial^\beta h_{\beta\alpha}\\
&+\tfrac{8}{9}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} - \tfrac{2}{9}\partial_\mu h\partial_\nu h - \tfrac{4}{3}\partial_\mu h^{\alpha\beta}\partial_\alpha h_{\beta\nu}\\
&-\tfrac{4}{3}\partial_\nu h^{\alpha\beta}\partial_\alpha h_{\beta\mu} + \tfrac{2}{3}\partial_\mu h_\nu{}^\alpha\partial^\beta h_{\beta\alpha} + \tfrac{2}{3}\partial_\nu h_\mu{}^\alpha\partial^\beta h_{\beta\alpha}\\
&+\tfrac{1}{6}\partial_\mu h\partial^\alpha h_{\alpha\nu} + \tfrac{1}{6}\partial_\nu h\partial^\alpha h_{\alpha\mu} - \tfrac{1}{6}\partial^\alpha h\partial_\mu h_{\nu\alpha}\\
&-\tfrac{1}{6}\partial^\alpha h\partial_\nu h_{\mu\alpha}\\
&+g_{\mu\nu}[h^{\alpha\beta}(\tfrac{1}{9}\partial^2 h_{\alpha\beta} - \tfrac{2}{3}\partial_\beta\partial^\lambda h_{\lambda\alpha} + \tfrac{1}{2}\partial_\alpha\partial_\beta h\\
&+\tfrac{1}{6}g_{\alpha\beta}\partial^\lambda\partial^\rho h_{\lambda\rho} - \tfrac{1}{9}g_{\alpha\beta}\partial^2 h)]\\
&+h_\mu{}^\alpha(\tfrac{1}{3}\partial^2 h_{\nu\alpha} - \tfrac{2}{3}\partial_\nu\partial^\beta h_{\beta\alpha} - \tfrac{1}{6}\partial_\nu\partial_\alpha h + \tfrac{2}{3}\partial_\alpha\partial^\beta h_{\beta\nu})\\
&+h_\nu{}^\alpha(\tfrac{1}{3}\partial^2 h_{\mu\alpha} - \tfrac{2}{3}\partial_\mu\partial^\beta h_{\beta\alpha} - \tfrac{1}{6}\partial_\mu\partial_\alpha h + \tfrac{2}{3}\partial_\alpha\partial^\beta h_{\beta\mu})\\
&+h^{\alpha\beta}(\tfrac{2}{3}\partial_\mu\partial_\alpha h_{\beta\nu} + \tfrac{2}{3}\partial_\nu\partial_\alpha h_{\beta\mu} - \tfrac{1}{9}\partial_\mu\partial_\nu h_{\alpha\beta} - \tfrac{4}{3}\partial_\alpha\partial_\beta h_{\mu\nu})\\
&-\tfrac{1}{6}h_{\mu\nu}\partial^2 h + \tfrac{1}{6}h\partial^2 h_{\mu\nu} - \tfrac{1}{6}h\partial_\mu\partial^\alpha h_{\alpha\nu} - \tfrac{1}{6}h\partial_\nu\partial^\alpha h_{\alpha\mu}\\
&+\tfrac{1}{9}h\partial_\mu\partial_\nu h,\end{aligned} \tag{11}$$

and $\theta_{\mu\nu}$ has the properties

$$\partial^\mu\theta_{\mu\nu} = 0 \qquad \theta_{\mu\nu} = \theta_{\nu\mu} \qquad \theta_\mu{}^\mu = 0. \tag{12}$$

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